Journal of Computing and Information Technology - CIT 18, 2010, 1, 45–52
doi:10.2498/cit.1001416

Refined Deterministic Algorithm for Biplane Construction

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This paper introduces a way of constructing biplanes that uses "filters" in a step prior to an exhaustive computer search, with the objective to make the biplane construction more efficient. The achieved deterministic algorithm classifies biplanes of order 7 and smaller. When performing the search using some additional presumptive conditions, two biplanes of order 9 ($k = 11$) were constructed and a particular inner regularity in the biplane's structure has been established.

Keywords: combinatorial design, biplane, exhaustive search, filtering, algorithm complexity

1. Introduction

A *biplane* is a symmetric 2-$(v, k, 2)$ design, i.e. an incidence structure B consisting of a set P of v *points* and of a set B of v *blocks*, so that each block contains k points and any two distinct blocks intersect in two points. Dual conditions are true as well — every point is incident with k blocks and every pair of points with two blocks. The parameters v and k are not independent, since it can be proven that the following equation holds for them: $2(v-1) = k(k-1)$. Until now, seventeen biplanes with seven different triples of parameters have been known. Among them, biplanes with 79 points are not completely classified, but two examples, possessing additional symmetries, are known [1]. Biplanes are structures of particular interest because of their highly regular structure, which follows from their already mentioned definition properties. The natural solution space size of biplanes increases extremely with the number of points v; this effect is usually called the *combinatorial explosion*. An additional difficulty while constructing biplanes is a very small number of found structures (results) in the solution space (they seem to appear very rare).

We have represented the biplane by its incidence matrix throughout this paper.

Definition 1. *Let* B *be a biplane with its point set* $P = \{p_1, \ldots, p_v\}$ *and block set* $B = \{b_1, \ldots, b_v\}$. *The incidence matrix* $M = [m_{ij}]$ *of* B *is a* $v \times v$ $0-1$ *matrix, the elements of which are defined by*

$$m_{ij} = \begin{cases} 1, & \text{if} \quad p_j \in b_i \\ 0, & \text{if} \quad p_j \notin b_i \end{cases} .$$

According to the biplane properties, $M = [m_{ij}]$ contains exactly k 1's in every row (which is in this representation a binary v-dimensional vector) as well as in every column. Due to the intersection conditions, every two distinct rows both contain 1's in exactly 2 columns (or equivalently, the "scalar product" of these binary vectors equals to 2) and dually, every two distinct columns have 1's on the same position twice. This paper addresses a way of biplane construction by means of an exhaustive search of appropriate 0-1 matrices. Classification algorithms of similar kind are nicely described in [4], [5] and [8]. In order to maximize the range of our algorithm, we use a filtering procedure of candidates for rows of the incidence matrix. While doing so, we have created a strategy for finding an optimal number of vectors in the filter. Our own custom C programs were written in order to implement all mentioned ideas.

Two 0-1 matrices M_1 and M_2 of the same dimension $m \times n$ are called *isomorphic*, if there is a permutation of rows and columns of M_1, which, when performed on it, results with M_2.

The notion of an isomorphism can be naturally implemented on incidence matrices of biplanes. It is of crucial importance to avoid constructing isomorphic copies of structures during the construction procedure as early as possible. Several different implementations on how to avoid constructing isomorphic copies of combinatorial configurations can be found in [11], [12] and [14].

In order to point out the importance of listing and classifying all combinatorial designs for given parameters, in particular biplanes, we remind the reader on the natural commonly used connection between designs and codes. Namely, one may look on the columns of an incidence matrix as on codewords of a block code over a finite field F_p of prime order p. Unfortunately, if p doesn't divide k-λ, the rank of the incidence matrix is maximal, which means that the columns are linearly independent and span the whole space F_p^v, so the corresponding code is trivial and of no importance. Hence, when p equals to 2, the most interesting case for applications in coding theory of all biplanes that will be constructed in this paper, are three biplanes of order 4, which lead to nice binary linear codes. For details the reader is referred to [2].

2. Space Size Reduction

Since we want to construct incidence matrices of order v, row by row, possessing k ones in each row, the natural size of our search space equals $\binom{v}{k}^v$. One of the main tasks is to shrink that number.

Proposition 1. *Let* B *be a biplane with parameters 2-(v,k,2) and let M=[m_{ij}] be its incidence matrix. Then first k rows of M can be uniquely determined, up to isomorphism. We shall call these beginning k rows base vectors.*

Proof: It is obvious that the first row can be chosen uniquely (up to isomorphism) as $11 \cdots 100 \cdots 0$, starting with k ones and ending with $v-k$ zeros. Note that the first k and the last $v-k$ columns remain mutually equivalent. Without losing generality, we may assume that the first point lies on the first k blocks, so put a 1 on the first position in all k rows. So, we have filled the upper-left frame of our $k \times v$ incidence matrix. The second row can now be chosen as $\underbrace{1}_{1}\underbrace{10\cdots0}_{k-1}\underbrace{1\cdots1}_{k-2}\underbrace{0\cdots0}_{v-2k+2}$ since the intersection with the first row has to be 2. We may fill free positions of the second columns with zeros now, as the first and the second column already have scalar product 2. The third row is again uniquely given as $\underbrace{1}_{1}\underbrace{010\cdots0}_{k-1}\underbrace{10\cdots0}_{k-2}\underbrace{1\cdots1}_{k-3}\underbrace{0\cdots0}_{v-3k+5}$. The free entries of the third column can be filled with zeros because of the same arguments. We proceed in a similar way and obtain for the *kth* row the following uniquely determined binary vector: $\underbrace{1}_{1}\underbrace{0\cdots01}_{k-1}\underbrace{0\cdots01}_{k-2}\underbrace{0\cdots01}_{k-3}\cdots\underbrace{01}_{2}\underbrace{1}_{1}$ □

Figure 1 illustrates the base vectors in case of a biplane 2-(16,6,2); we believe that this example gives a good graphical explanation of the construction of the base vectors for any biplane, as proven in the previous proposition.

1	1	1	1	1	1										
1	1					1	1	1	1						
1		1				1				1	1	1			
1			1				1			1			1	1	
1				1				1			1		1		1
1					1				1			1		1	1

Figure 1. Base vectors of the biplane 2-(16,6,2).

Determining the first k rows of our incidence matrix reduces significantly the solution space size, at least from $\binom{v}{k}^v$ to $\binom{v}{k}^{v-k}$. Furthermore, the other v-k variable rows must have exactly two 1's among the first k positions, because of the intersection condition with the first vector. That fact leads to a restriction of the upper bound for the complexity to $\left[\binom{k}{2}\binom{v-k}{k-2}\right]^{v-k}$. Finally, the first element of every variable row must be 0, since the first column already has k 1's in the base vectors; hence, the solution space size can be reduced at least to $\left[\binom{k-1}{2}\binom{v-k}{k-2}\right]^{v-k}$.

Even more can be said about the first k columns of the incidence matrix of order v which we want to construct. Using exactly the dual arguments to those from the proof of Proposition 1, one gets its dual statement, which ensures the unique content of the first k columns. Note that the part of the incidence matrix which we have given uniquely is symmetric. We illustrate this situation in Figure 2. The solution space size is now reduced to $\begin{pmatrix} v-k \\ k-2 \end{pmatrix}^{v-k}$.

1	1	1	1	1	1										
1	1					1	1	1	1						
1		1				1				1	1	1			
1			1				1			1			1	1	
1				1				1			1		1		1
1					1				1			1		1	1
	1	1													
	1		1												
	1			1											
	1				1										
		1	1												
		1		1											
		1			1										
			1	1											
			1		1										
				1	1										

Figure 2. First 6 rows and first 6 columns of the biplane 2-(16,6,2) can be determined uniquely.

The up to isomorphism unique determination of the first k rows and columns leads further to a natural decomposition of the incidence matrix M which we want to construct in a block matrix of order k-1, as shown in Figure 3.

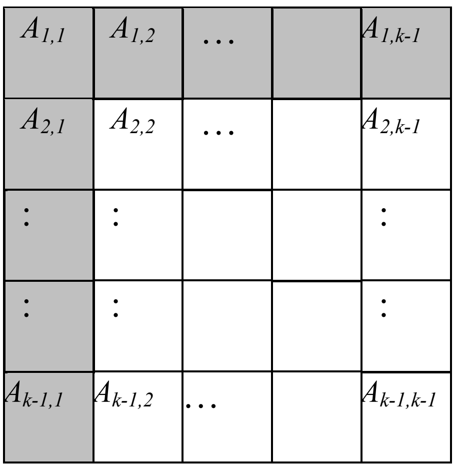

Figure 3. Incidence matrix decomposition.

Here, $A_{2,2}$ is a matrix of order k-2, whereas $A_{k-1,k-1}$ is of order 1. Once the matrix M has been decomposed in such a way, its rows and columns satisfy some additional properties. The conditions on rows can be easily derived from the intersection properties with the first uniquely constructed k rows (base vectors), while the conditions on columns can be achieved applying dual arguments. For example, each row of $A_{2,2}$ must have exactly one element equal to 1, because of the intersection with the second base vector and the fact that the second column of $A_{2,1}$ is filled only with 1's.

Proposition 2. *Let M=[m_{ij}] be an incidence matrix of a biplane with parameters 2-(v,k,λ) being decomposed as described above. Then for the entries of M holds:*

i) The diagonal submatrix $A_{2,2}$ has in each row and in each column exactly one element equal to 1, whereas the submatrices $A_{3,3}, \ldots, A_{k-1,k-1}$ have zero or one 1's in each row and each column.

ii) Submatrices $A_{i,2}$, i>2, have in each row exactly two 1's; submatrices $A_{i,3}$, i>3, have one or two 1's in each row; submatrices $A_{i,j}$, i,j>3, $i \neq j$, have in each row zero, one or two 1's.

ii') Submatrices $A_{2,j}$, j>2, have in each column exactly two 1's; submatrices $A_{3,j}$, j>3, have one or two 1's in each column; submatrices $A_{i,j}$, i,j>3, $i \neq j$, have in each column zero, one or two 1's.

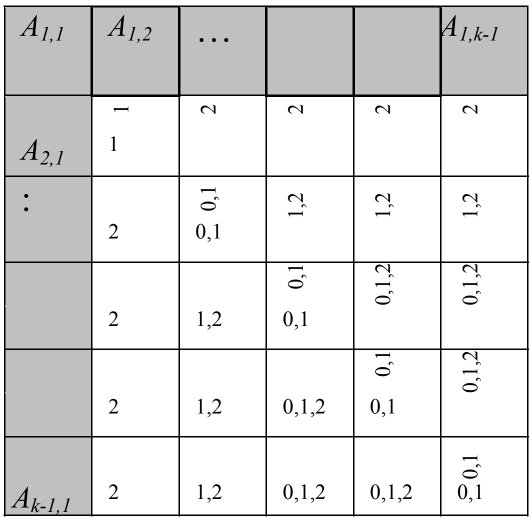

Figure 4. Admissible number of 1's in M.

In the next figure, we give a scheme of the decomposed incidence matrix fulfilling these conditions.

Knowing the first k rows (the base vectors) without loss of generality, the next step in biplane construction is to generate all possible vectors — row candidates for the variable part (rows $k+1$ to v) of the incidence matrix and isolate those candidates which intersect every base vector in two points. Note that we have used the k base vectors as a filter for the *vector candidates* for the variable rows.

We want to give a simple mathematical argument which avoids the use of the duality for the construction of the first k columns. The equation between the parameters v and k of a biplane can be rewritten as $\binom{k-1}{2} = v-k$, which gives a hint to take all 2-element subsets as incidences in columns 2 to k. Indeed, since the points corresponding to these columns lie together all on the very first block, this is the only possibility (as we already know!). According to this, all filtered vector candidates for the rows $k+1$ to v can be, naturally, separated into groups which contain vectors with the same beginning k-tuple. Vectors in each particular group are admissible candidates for a corresponding row of the incidence matrix. Table 1 shows results obtained for known biplanes. As we could convince ourselves, each group of filtered row candidates contains the same number of vectors (*numvect*), which is a natural consequence of the fact that the same condition had to be checked against the base rows. It can be noticed that, already at this step, the biplane (7,4,2) is constructed and classified.

(v, k, λ)	*numvect* in variable part	*numvect* in a row
(7,4,2)	3	1
(11,5,2)	12	2
(16,6,2)	70	7
(37,9,2)	30016	1072
(56,11,2)	3026655	67259
(79,13,2)	438263364	6640354

Table 1. Number of vector candidates for each row that intersect the base vectors in 2 points.

3. Classification of Biplanes of Small Orders

Once we have constructed the base k vectors of an incidence matrix and successfully generated v-k groups of vector candidates, which all intersect the base vectors in two points, the solution space size reduces significantly. The classification of biplanes with v points can now be done by an exhaustive search of vector candidates.

Since an evident approach of using backtracking algorithm wouldn't be satisfactorily effective, this paper introduces another vector filtering prior to an exhaustive search. At the first step of such an algorithm, a proper combination of f vectors ought to be chosen from the vector candidates for rows $k+1, k+2, \ldots, k+f$, satisfying the condition that every pair of these candidates intersects in 2 points (f is the number of vectors in the *filter*). After that, vector candidates for the other v-k-f rows are filtered, meaning in particular that only those vectors survive that intersect with every vector from the chosen filter in 2 positions. Finally, an exhaustive search of the *survived vector candidates* for the v-k-f rows can be made. Figure 5 presents a pseudocode of the described algorithm. Note that the most time consuming operation in this

```
Algorithm ConstructBiplane(v,k,λ,f)
generate base vectors
generate row vector candidates and separate them
        into v-k groups
while filter exists
  BuildFilter:
     find next admissible combination of f
        vectors, from (k+1)th to (k+f)th row
  Filtering:
     for i = k + f + 1 to v
        in ith row select only vectors that intersect
           properly with the filter
  Exhaustive search:
     while inc. matrix exists
        find next admissible combination of v-k-f
           selected vector candidates,
        from (k+f+1)th row to vth row
           inc.matrix = base vectors + filter +
              admissible combination of v-k-f vectors
```

Figure 5. Pseudocode of the developed algorithm.

algorithm is the determination of intersection of two vectors. Throughout our implementations, we represent the vectors as bit sequences (the parameters of which have been optimized), which enables us to use bitwise operators for calculations (see [8]).

Before using this algorithm, the number f of vectors in the filter was optimized for all biplane parameters for which a complete classification is known. Table 2 shows these results in the case of biplane 2-(37,9,2). The number of basic operations (intersection determination between two vectors, *numiter*) for one local exhaustive search (for the first vector chosen from the group of vector candidates for the 10*th* row) is presented and compared with f. Also, the number of basic operations for both filtering and exhaustive search parts of the algorithm is measured and compared (Figure 6). It can be seen that two kinds of operations give the basic operation minimum when the number f of vectors in the filter is equal to 3.

When the algorithm *ConstructBiplane* was implemented, the classification of biplanes with 11, 16 and 37 points was done. After generating all incidence matrices (*numincmat*), for given biplane's parameters, non-isomorphic matrices

	numiter (for the 1st 10th row candidate)		
f	exhaustive search	filter	sum
1	1684894714	28944	1684923658
2	459044053	2608785	461652838
3	117721020	64163364	181884384
4	28719700	396435510	425155210

Table 2. Optimal f for the biplane (37,9,2).

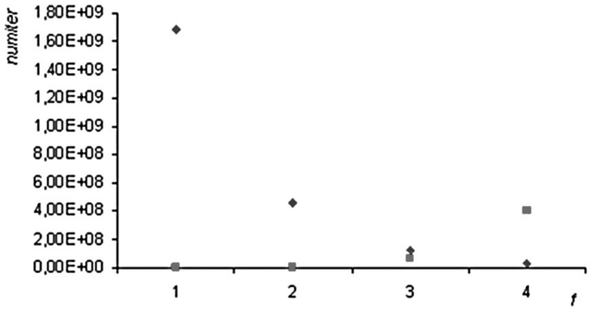

Figure 6. Relationship between *numiter* in filtering and exhaustive search; here in case of biplane (37,9,2).

were extracted by a computer program *incfilter* [7]. Also, the automorphism group orders of these isolated non-isomorphic matrices were determined. Table 3 presents obtained results. As it was known, there is one biplane of order 3, three biplanes of order 4 and four biplanes of order 7 (see [13]).

$\mathcal{B}$	*numincmat*		$\mid Aut\ (\mathcal{B}) \mid$
	total	up to isom.	
(11,5,2)	2	1	660
(16,6,2)	46	3	384, 768, 11520
(37,9,2)	306720	4	54, 333, 1512, 1512

Table 3. Biplanes classified by the algorithm *ConstructBiplane*.

4. Construction of Biplanes of Order 9 and Biplane's Inner Regularities

Examining the set of all constructed incidence matrices of biplanes with parameters (16,6,2) and (37,9,2) more carefully, two interesting features could be observed. Namely, for some of these matrices the following statements hold.

i) Each element on the main diagonal is equal to 1 ($m_{k+1,k+1} = m_{k+2,k+2} = \ldots m_{v,v} = 1$).

ii) The matrix is symmetric with respect to the main diagonal.

When we combine the first feature to the decomposition form of the incidence matrix described by Proposition 2, we get strong conditions on the entries of the incidence matrix under construction which are again very useful in shrinking the size of the solution space. We shall implement this idea without going further into details.

Since the algorithm *ConstructBiplane* was not able to solve the construction and classification problem for the biplane of the next order, having parameters 2-(56,11,2), we decided to make a partial exhaustive search, using some additional assumptions on the biplane structure. According to the observed matrix features mentioned above, we assumed the existence of a biplane for (56,11,2) possessing an all-1 main diagonal. When row candidates which fulfill this additional criterion were generated, the partial exhaustive search using the introduced filter with $f = 4$ vectors could be finished successfully and it resulted with two biplanes of order 9. Among 312 constructed incidence matrices, two were non-isomorphic, with automorphism groups of order 144 and 80640 (Table 4).

$\mathcal{B}$	*numvect* in row	*numincmat*	$\lvert Aut\ (\mathcal{B})\rvert$
(16,6,2)	1	1	11520
(37,9,2)	70	480	1512,1512
(56,11,2)	3507	> 312	144, 80640

Table 4. Biplanes with 1's on the main diagonal.

Except for the two biplanes of order 9, there exist one biplane of order 4 and two biplanes of order 7 (duals, with automorphism group of order 1512) having this property. In the case of the parameter set (37,9,2), the optimal number f of vectors in the filter was 2, whereas in the case of (56,11,2), it was 4.

When constructing biplanes of order 7, in case of having the introduced additional criterion (all-1 main diagonal), one notable regularity of the incidence matrix structure can be observed. For each row candidate for the 10*th* row (representing in fact a 1-filter), there are either 24 or 30 row candidates for the 11*th* row which survive the filter. Moreover, in the case when 24 row candidates survive the 1-filter for the 11*th* row, 4 matrices were constructed and in the other case, when 30 candidates survive, 24 matrices were constructed. An analogue regularity has been verified also in case of biplanes of order 9, to be more precise, on those biplanes on 56 points which we were able to construct. Our results are presented in Table 5 and Table 6. Having in mind this regularity we have just described, it makes sense, for any biplane with parameters (v,k,λ), to call the number of vectors for the $(k+j)th$ row $(j = 2,\ldots,v-k)$, which survive the 1-filter from the $(k+1)th$ row, the *1-filter $(k+j)$-row track* (or shortly, when all the included parameters are obvious, the *track*).

track	frequency	*numincmat*	$\lvert Aut(\mathcal{B})\rvert$
30	10	24	1512
24	60	4	1512

Table 5. Inner regularity of biplanes of order 7.

Having now in mind, that only for some $(k+1)$-row candidates (1-filters) a biplane can be constructed, it would be more efficacious if we could perform further our search for completing the $(k+1)\times v$ 0-1 matrix to an incidence matrix on v points, only for particularly selected, "promising" 1-filters. For example, as shown in Table 6, we get positive results for our construction only for those 12*th* row candidates for which the 13-row track equals 1098 or 1116. So, we would like to know in advance that it makes sense to continue the construction of the incidence matrix only for some12*th* row candidates. But, how to predict which 1-filters are *good filters* in that sense?

Making a test for a biplane with parameters (56,11,2), as shown in Table 7, a nice regularity appears. Namely, for some of the 1-filters, their track keeps its value constant for many succeeding rows. This fact gives us a hint for our criterion for the choice of a good 1-filter.

Furthermore, for any j, we can admit the set of $(k+j)th$ row candidates to become row candidates for the first variable $(k+1)th$ row, since it is possible (once fixing the first k positions in every variable row) to permute variable rows of the incidence matrix without loss of generality; this leads to a generalization of the term good filter on the whole row candidate vector set. Selecting only good vectors in every row for the row candidates, we are able to construct biplanes more efficiently. During such selective constructions, we have noticed a particular

track	frequency	*numincmat*	$\lvert Aut(\mathcal{B})\rvert$
1059	420	0	–
1098	2520	4	144
1116	315	136	144,80640
1131	252	0	–

Table 6. Inner regularity of constructed biplanes of order 9.

symmetry in the obtained incidence matrices. In case of the biplane of order 9 construction, for one particular "type" of good vectors, we have obtained symmetric matrices with respect to the main diagonal and with the property that submatrices $A_{2,3}$ and $A_{3,2}$ are skew-symmetric. In an analogue selective construction of the biplane $(37,9,2)$, only a symmetry of submatrices $A_{2,3}$ and $A_{3,2}$ could be verified.

row	ordinal number of vector from $(k+1)$st row				
$(k+1)$	424.	3006.	500.	838.	12.
			track		
$(k+2)$	1059	1059	1098	1098	1116
$(k+3)$	1059	1131	1098	1098	1116
$(k+4)$	1131	1131	1098	1098	1116
$(k+5)$	1059	1131	1098	1098	1116
$(k+6)$	1059	1059	1098	1098	1116
$(k+7)$	1131	1059	1098	1098	1116
$(k+8)$	1131	1059	1098	1098	1116
$(k+9)$	1059	1059	1098	1098	1116
...					
$(k+17)$	1098	1059	1098	1098	1116
$(k+18)$	1035	977	1123	1123	1160
$(k+19)$	977	977	977	1037	1160

Table 7. Good filters in the case of a biplane (56,11,2).

5. Conclusion

In this paper we have developed combinatorial algorithms for constructing symmetric designs with parameters $(v,k,2)$ in full generality, for any number of points v. After being able to reduce the solution space significantly without loss of generality, by fixing some parts of the incidence matrix, which represents the structure we are looking for, the deterministic algorithm *ConstructBiplane* was created and implemented. Our algorithm makes use of a filtering procedure to increase the algorithm range.

Once having the filter size optimized, an exhaustive search was carried out and biplanes with up to 37 points were classified (Table 3). Since the construction of a biplane with 56 points was out of range for this algorithm, an additional criterion introducing an all-one diagonal for the incidence matrix was introduced and a partial exhaustive search was done. This criterion came to our mind when inspecting the biplanes which we had classified before. After finishing this approach successfully, two biplanes with 56 points were constructed, with automorphism group orders 144 and 80640.

Even with that additional criterion, we could not find any biplane on 79 points, although they are known to exist. Further presumptive strategies led us to particular selections of candidates for rows of the incidence matrix, all leading only to the already mentioned examples.

To speed up the algorithm run time as much as possible, our algorithm uses bitwise operators; still, that was not decisive for any of our constructions, as we could not move the frontiers of parameters in question.

Algorithm's behavior has confirmed typical biplane characteristics: the solution space size grows enormously with the number of points v, while positive results within this space happen to be extremely rare.

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Received: January, 2009
Accepted: December, 2009

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